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Photon emission by electrons and positrons created in a strong slowly rotating magnetic field

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Abstract. We study the electrodynamic process in which a photon is emitted together with an e^-–e^+ pair in the presence of a strong slowly rotating magnetic field. In particular, the spectrum of photons produced in this way is calculated starting from an effective Lagrangian that allows at tree level for the process itself. The magnetic field strengths we have in mind are $\sim 10^{14}$ G in such a way that, although our model is an oversimplified version of the real physical situation, the results can be applied only in some particular astrophysical scenarios (magnetars, massive black holes).

1 Introduction

There is today much indirect evidence that overcritical [$\gg B_{\text{cr}} = m^2c^3/(\hbar e) \simeq 4.4 \times 10^{13}$ G] magnetic fields may be present around very highly magnetized and rotating neutron stars called magnetars [1–3]. Also, numerical simulations predict that around massive black holes surrounded by an accretion disk such strong magnetic fields may be present too [4]. For these reasons, studying the electrodynamic processes that can occur in the presence of such strong fields is not a purely theoretical exercise but it can checked by (at least indirect) experimental observations. In the present paper we want to study the production of a photon together with an e^-–e^+ pair in the presence of a strong slowly rotating magnetic field. This work follows other papers where the production of electrons, positrons [5–7] and photons [8] in the presence of strong slowly varying magnetic fields in various configurations has been studied. In particular, in [8] the spectrum was obtained of the photons produced after the annihilation of e^-–e^+ pairs previously created in a strong slowly rotating magnetic field. In the present work we want to calculate the spectrum of the photons emitted through a different mechanism. In fact, it is well known that an electron or a positron in the presence of a constant and uniform magnetic field can emit photons as synchrotron radiation [9]. Nevertheless, as we have said before, the process we want to consider here is quite different. In fact, we want to study the whole process consisting both in the creation of an e^-–e^+ pair and in the electromagnetic emission of a photon by one of the particles making the pair. In particular, it is worth pointing out that the creation of the pair is possible only because of the time variation of the background magnetic field and of the consequent presence of an induced electric field [10]. In this respect, the time variation of the magnetic field is an essential ingredient of the calculation and distinguishes it from the others already done in the literature [see e.g. [11] and references therein or the less recent review [12]]. Incidentally, the case of a rotating magnetic field is relevant because the production of e^-–e^+ pairs (and then of photons) is much more efficient than the production in the presence of a magnetic field varying only in strength [7]. Moreover, this particular time evolution makes the mathematical treatment of the problem easier.

Apart from its own relevance, a possible application of these calculations is the study of the spectrum of gamma-ray bursts [13]. Although the standard fireball model [14] explains many features of gamma-ray bursts such as the general form of the energy spectra or the fact that the photon radiation is highly polarized [15], the mechanism that primes the formation of the fireball (made essentially of electrons, positrons and photons) is not completely clear. Nevertheless, it seems almost sure that the fireball is produced near forming neutron stars or black holes surrounded by a rotating torus of debris as a consequence of a catastrophic event such as a supernova explosion [16]. Now, we are aware that the theoretical framework in which our calculations are carried out is a very simplified version of the real astrophysical scenario where gamma-ray bursts are produced. In particular, many macroscopic and collective aspects that are present in the real physical environment have been neglected. From this point of view, our model has to be considered as a "toy model" that tries to reproduce qualitatively some experimental features of the very complicated phenomenon of gamma-ray bursts and, in particular, of their energy spectra. Concerning the applicability of the following calculations to gamma-ray bursts,

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another word of caution must be said. In fact, a gamma-ray burst is produced in a region where the local densities of electrons (positrons) are very high (there are $\sim 10^{54}$ of these particles in a volume with typical length $\sim 10^6$ cm). In these extreme conditions the probability that the photons emitted through the mechanism at hand reach directly an observer at infinity may be low. In fact, it is likely that they interact with the existing electrons and positrons in such a way that the photon spectrum seen at infinity could be different from that calculated here (see Sect. 3).

The plan of this paper is given below. Our theoretical starting point is the Lagrangian density of QED in the presence of an external rotating magnetic field. The fact that the magnetic field is purely rotating allows us to build an effective time-independent Lagrangian density that takes into account the rotation of the external field through additional interaction terms proportional to the rotational frequency of the magnetic field (Sect. 2). In this way, since, as we will see, the magnetic field can be assumed to be slowly rotating, the ordinary perturbation theory has been used to calculate the spectrum of the emitted photons (Sect. 3). As we conclude in Sect. 4, although the simplicity of our theoretical model with respect to the real physical situation, qualitative features of experimental spectra of gamma-ray bursts are reproduced such as the linear dependence on the inverse of the photon energy in the low energy region or the presence of a "break" energy around which the spectrum shows two different behaviors.

To conclude this Introduction we want to mention some notational points: the signature of the Minkowski spacetime is assumed to be $+---$ and, while Greek indices run from 0 to 3, Latin ones run from 1 to 3. Finally, as usual, natural units ($\hbar = c = 1$) are used throughout.

2 Theoretical model

The process to be studied concerns the production of electrons, positrons and photons in the presence of a strong rotating background magnetic field. A good theoretical starting point is the Lagrangian density $\mathcal{L}(\psi, \partial_\mu\psi, \bar{\psi}, A^{(r)}_\mu, \partial_\nu A^{(r)}_\mu, \mathbf{r}, t)$ of QED in the presence of an external electromagnetic field. If $A_\mu(\mathbf{r}, t)$ is the four-potential describing the external electromagnetic field, then

$$\begin{aligned}
&\mathcal{L}(\psi, \partial_\mu\psi, \bar{\psi}, A^{(r)}_\mu, \partial_\nu A^{(r)}_\mu, \mathbf{r}, t)\\
&= \bar{\psi}(\mathbf{r}, t)\left\{\gamma^\mu\left[\mathrm{i}\partial_\mu + eA_\mu(\mathbf{r}, t) + eA^{(r)}_\mu(\mathbf{r}, t)\right] - m\right\}\psi(\mathbf{r}, t)\\
&-\frac{1}{4}F^{(r)}_{\mu\nu}(\mathbf{r}, t)F^{(r)\mu\nu}(\mathbf{r}, t) \qquad (1)
\end{aligned}$$

where the radiation field

$$A^{(r)}_\mu(\mathbf{r}, t) = \left[\varphi^{(r)}(\mathbf{r}, t), -\mathbf{A}^{(r)}(\mathbf{r}, t)\right]$$

is assumed in the Coulomb gauge

$$\varphi^{(r)}(\mathbf{r}, t) = 0, \qquad (2)$$

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(r)}(\mathbf{r}, t) = 0, \qquad (3)$$

and where

$$F^{(r)}_{\mu\nu}(\mathbf{r}, t) = \partial_\mu A^{(r)}_\nu(\mathbf{r}, t) - \partial_\nu A^{(r)}_\mu(\mathbf{r}, t). \qquad (4)$$

In (1) the two terms proportional to $F_{\mu\nu}(\mathbf{r}, t)F^{\mu\nu}(\mathbf{r}, t)$ and to $F_{\mu\nu}(\mathbf{r}, t)F^{(r)\mu\nu}(\mathbf{r}, t)$ with $F_{\mu\nu}(\mathbf{r}, t) = \partial_\mu A_\nu(\mathbf{r}, t) - \partial_\nu A_\mu(\mathbf{r}, t)$ have been omitted because they do not give any significant contribution to the equations of motion of the fields $\psi(\mathbf{r}, t)$ and $A^{(r)}_\mu(\mathbf{r}, t)$ and to the process we want to study.

In our case the external electromagnetic field is the magnetic field created by an astrophysical compact object and it can be safely assumed to be uniform in the microscopical length scale (of the order of $\lambda\!\!\!^{-}_c = 1/m$) in which an e^-–e^+ pair is created[1]. Finally, if we also assume the magnetic field $\mathbf{B}(t)$ to be static before $t = 0$ and purely rotating in the y–z plane after $t = 0$ it can be written as

$$\mathbf{B}(t) = \begin{cases} B\,(0, \sin\Omega t, \cos\Omega t) & \text{if } t \geq 0,\\ B\,(0, 0, 1) & \text{if } t < 0,\end{cases} \qquad (5)$$

and the four-potential $A_\mu(\mathbf{r}, t)$ can be chosen in the form $A_\mu(\mathbf{r}, t) = [0, -\mathbf{A}(\mathbf{r}, t)]$ with

$$\mathbf{A}(\mathbf{r}, t) = -\frac{1}{2}\left[\mathbf{r}\times\mathbf{B}(t)\right]. \qquad (6)$$

We pointed out in (5) that the magnetic field is static before an arbitrary time set equal to zero.[2] The fact that the magnetic field has always strength B for times $t < 0$ must be clarified because a sudden appearance of the magnetic field would imply huge induction effects. Actually, from a theoretical point of view, we tacitly assume that at very large times in the past the magnetic field grew adiabatically from zero to B without changing its direction. Now, also in this phase there is a production of particles (electrons, positron and photons) but, as we have checked in [7], a changing-direction magnetic field primes much more efficient pair-production mechanisms than a magnetic field changing only in strength [in general the production probabilities in the first case are $(B/B_{\mathrm{cr}})^{3/2} \gg 1$ times the corresponding probabilities in the second one]. In this framework, we have tacitly assumed that we could neglect the presence of the particles created in this phase with respect to those that will be created at $t \geq 0$ as a consequence of the rotation of the magnetic field. In particular, in the astrophysical environment we sketched in the Introduction the instant zero could represent, for example, the time when the supernova explosion begins.

Now, the Lagrangian density (1) is not in the most suitable form for the physical scenario we want to describe. In fact, as it stands it would be suitable for dealing with a weak external magnetic field because the usual perturbation theory could be used, while, as we have said in the

[1] Since in the present paper we also deal with photons, we point out that the previous assumption about the uniformity of the magnetic field is valid only for photons with energy ω such that $\Omega \ll \omega$ [8].

[2] It can be shown that a purely rotating magnetic field from $-\infty$ to ∞ would not give any particle production.

Introduction, we are dealing with a strong magnetic field. Nevertheless, we also know that the temporal evolution scale of the macroscopic magnetic field (5) can be assumed to be much larger than that of the typical times in which the electrons and positrons are created, that is, $\Omega \ll m$ [7]. For this reason, in the following, our goal is to manipulate the Lagrangian density (1) in order to write it in a form that allows us to exploit the "adiabatic" temporal evolution of the magnetic field (5)[3]. Firstly, we perform the time-depending rotation

$$\begin{aligned}\mathbf{r}' &\equiv (x', y', z') \\ &= (x, y\cos\Omega t - z\sin\Omega t, y\sin\Omega t + z\cos\Omega t),\end{aligned} \tag{7}$$

where, for later notational simplicity, we have not indicated the dependence of the variables $\mathbf{r}'$ on t. As a consequence, the spinor field $\psi(\mathbf{r}, t)$ and the four-vector field $A^{(r)}(\mathbf{r}, t) = [A^{(r)0}(\mathbf{r}, t), \ldots, A^{(r)3}(\mathbf{r}, t)]$ transform as

$$\psi'(\mathbf{r}', t) = \exp\left(-\mathrm{i}\frac{\sigma_x}{2}\Omega t\right)\psi(\mathbf{r}, t), \tag{8}$$

$$A^{(r)\prime}(\mathbf{r}', t) = \exp\left(-\mathrm{i}S_x\Omega t\right)A^{(r)}(\mathbf{r}, t), \tag{9}$$

where the matrices σ_x and S_x are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad S_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \end{pmatrix}. \tag{10}$$

We point out that, although σ_x and S_x are two (4×4) matrices, they act on two different spaces: the first one acts on the spinor space and the second one acts on the four-vector space labeled by the Lorentz indices $\{0, \ldots, 3\}$.

The current density $j(\mathbf{r}, t) = \bar{\psi}(\mathbf{r}, t)\gamma\psi(\mathbf{r}, t)$ with $\gamma \equiv (\gamma^0, \ldots, \gamma^3)$ transforms under the rotation (7) as a four-vector; then [see (9)]

$$j(\mathbf{r}, t) = \exp\left(\mathrm{i}S_x\Omega t\right)j'(\mathbf{r}', t), \tag{11}$$

with $j'(\mathbf{r}', t) = \bar{\psi}'(\mathbf{r}', t)\gamma\psi'(\mathbf{r}', t)$. These previous equations can be exploited to rewrite the Lagrangian density (1) in terms of the primed variables and fields. In particular, it is evident from (9) and (11) that

$$\bar{\psi}(\mathbf{r}, t)\gamma^\mu\psi(\mathbf{r}, t)A^{(r)}_\mu(\mathbf{r}, t) = \bar{\psi}'(\mathbf{r}', t)\gamma^\mu\psi'(\mathbf{r}', t)A^{(r)\prime}_\mu(\mathbf{r}', t). \tag{12}$$

The transformation of the terms involving the external magnetic field are more complicated. In fact, from (5) and (6) we obtain

$$\begin{aligned}&\bar{\psi}(\mathbf{r}, t)\gamma^\mu\psi(\mathbf{r}, t)A_\mu(\mathbf{r}, t) \\ &= \bar{\psi}'(\mathbf{r}', t)\gamma^1\psi'(\mathbf{r}', t) \\ &\quad\times\frac{1}{2}\left[(y'\cos\Omega t + z'\sin\Omega t)B\cos\Omega t\right. \\ &\quad\left.-(z'\cos\Omega t - y'\sin\Omega t)B\sin\Omega t\right] \\ &\quad+\bar{\psi}'(\mathbf{r}', t)\left(\gamma^2\cos\Omega t + \gamma^3\sin\Omega t\right)\psi'(\mathbf{r}', t) \\ &\quad\times\frac{1}{2}(-x')B\cos\Omega t \\ &\quad+\bar{\psi}'(\mathbf{r}', t)\left(\gamma^3\cos\Omega t - \gamma^2\sin\Omega t\right)\psi'(\mathbf{r}', t)\frac{1}{2}x'B\sin\Omega t \\ &= \bar{\psi}'(\mathbf{r}', t)\gamma^\mu\psi'(\mathbf{r}', t)A'_\mu(\mathbf{r}'),\end{aligned} \tag{13}$$

where we introduced the four-potential $A'_\mu(\mathbf{r}') = [0, -\mathbf{A}'(\mathbf{r}')]$ with

$$\mathbf{A}'(\mathbf{r}') = -\frac{1}{2}\left(\mathbf{r}' \times \mathbf{B}'\right) \tag{14}$$

the vector potential corresponding to the static magnetic field $\mathbf{B}' = (0, 0, B)$.

Now, we want to see how the terms containing derivatives of the fields in the Lagrangian density (1) transform. We first transform separately the time derivatives $\partial\psi(\mathbf{r}, t)/\partial t$ and $\partial A^{(r)}(\mathbf{r}, t)/\partial t$. From (8) and (9) and by recalling that the variables $\mathbf{r}'$ actually depend on time [see (7)], we have

$$\begin{aligned}&\frac{\partial\psi(\mathbf{r}, t)}{\partial t} \\ &= \exp\left(\mathrm{i}\frac{\sigma_x}{2}\Omega t\right)\left[\mathrm{i}\Omega\mathcal{J}_x^{(1/2)\prime}\psi'(\mathbf{r}', t) + \frac{\partial\psi'(\mathbf{r}', t)}{\partial t}\right],\end{aligned} \tag{15}$$

$$\begin{aligned}&\frac{\partial A^{(r)}(\mathbf{r}, t)}{\partial t} \\ &= \exp\left(\mathrm{i}S_x\Omega t\right)\left[\mathrm{i}\Omega\mathcal{J}_x^{(1)\prime}A^{(r)\prime}(\mathbf{r}', t) + \frac{\partial A^{(r)\prime}(\mathbf{r}', t)}{\partial t}\right],\end{aligned} \tag{16}$$

where we introduced the one-particle total angular momentum operators

$$\mathcal{J}_x^{(1/2)\prime} = \frac{1}{\mathrm{i}}\left(y'\frac{\partial}{\partial z'} - z'\frac{\partial}{\partial y'}\right) + \frac{\sigma_x}{2}, \tag{17}$$

$$\mathcal{J}_x^{(1)\prime} = \frac{1}{\mathrm{i}}\left(y'\frac{\partial}{\partial z'} - z'\frac{\partial}{\partial y'}\right) + S_x. \tag{18}$$

By means of (15) and (16) it can be seen that

$$\begin{aligned}&\bar{\psi}(\mathbf{r}, t)\gamma^\mu\partial_\mu\psi(\mathbf{r}, t) \\ &= \bar{\psi}'(\mathbf{r}', t)\exp\left(-\mathrm{i}\frac{\sigma_x}{2}\Omega t\right)\gamma^0\partial_0\left[\exp\left(\mathrm{i}\frac{\sigma_x}{2}\Omega t\right)\psi'(\mathbf{r}', t)\right] \\ &\quad+\bar{\psi}'(\mathbf{r}', t)\gamma^1\partial'_1\psi'(\mathbf{r}', t) \\ &\quad+\bar{\psi}'(\mathbf{r}', t)\left(\gamma^2\cos\Omega t + \gamma^3\sin\Omega t\right) \\ &\qquad\times\left(\cos\Omega t\,\partial'_2 + \sin\Omega t\,\partial'_3\right)\psi'(\mathbf{r}', t) \\ &\quad+\bar{\psi}'(\mathbf{r}', t)\left(-\gamma^2\sin\Omega t + \gamma^3\cos\Omega t\right) \\ &\qquad\times\left(-\sin\Omega t\,\partial'_2 + \cos\Omega t\,\partial'_3\right)\psi'(\mathbf{r}', t) \\ &= \bar{\psi}'(\mathbf{r}', t)\gamma^\mu\partial'_\mu\psi'(\mathbf{r}', t) \\ &\quad+\mathrm{i}\Omega\bar{\psi}'(\mathbf{r}', t)\gamma^0\mathcal{J}_x^{(1/2)\prime}\psi'(\mathbf{r}', t),\end{aligned} \tag{19}$$

[3] Analogously to what we have said in the note 1, we also tacitly agree to restrict our attention to photons with energy ω such that $\Omega \ll \omega$ and only with this further assumption we are allowed to consider as adiabatic the time evolution of $\mathbf{B}(t)$.

and that

$$
\begin{aligned}
-\frac{1}{4}&F^{(r)}_{\mu\nu}(\mathbf{r},t)F^{(r)\mu\nu}(\mathbf{r},t)\\
&=-\frac{1}{4}F^{(r)\prime}_{\mu\nu}(\mathbf{r}',t)F^{(r)\prime\mu\nu}(\mathbf{r}',t)\\
&\quad -\mathrm{i}\Omega\frac{\partial A^{(r)\prime}_{\mu}(\mathbf{r}',t)}{\partial t}\mathcal{J}^{(1)\prime}_{x}A^{(r)\prime\mu}(\mathbf{r}',t)+O(\Omega^2),
\end{aligned}
\tag{20}
$$

where

$$
F^{(r)\prime}_{\mu\nu}(\mathbf{r}',t)=\partial'_{\mu}A^{(r)\prime}_{\nu}(\mathbf{r}',t)-\partial'_{\nu}A^{(r)\prime}_{\mu}(\mathbf{r}',t),
\tag{21}
$$

and where, by exploiting the adiabatic time evolution of the magnetic field (5), we neglected the terms proportional to Ω^2.

By collecting (12), (13), (19) and (20) and by performing the remaining trivial transformation of the mass term in the Lagrangian density (1), it can be written in terms of the primed variables and fields as follows:

$$
\begin{aligned}
&\mathcal{L}'(\psi',\partial'_{\mu}\psi',\bar{\psi}',A^{(r)\prime}_{\mu},\partial'_{\nu}A^{(r)\prime}_{\mu},\mathbf{r}')\\
&=\bar{\psi}'(\mathbf{r}',t)\left\{\gamma^{\mu}\left[\mathrm{i}\partial'_{\mu}+eA'_{\mu}(\mathbf{r}')+eA^{(r)\prime}_{\mu}(\mathbf{r}',t)\right]-m\right\}\\
&\qquad\times\psi'(\mathbf{r}',t)\\
&\quad-\frac{1}{4}F^{(r)\prime}_{\mu\nu}(\mathbf{r}',t)F^{(r)\prime\mu\nu}(\mathbf{r}',t)\\
&\quad+\mathrm{i}\Omega\bar{\psi}'(\mathbf{r}',t)\gamma^0\mathcal{J}^{(1/2)\prime}_{x}\psi'(\mathbf{r}',t)\\
&\quad-\mathrm{i}\Omega\frac{\partial A^{(r)\prime}_{\mu}(\mathbf{r}',t)}{\partial t}\mathcal{J}^{(1)\prime}_{x}A^{(r)\prime\mu}(\mathbf{r}',t)+O(\Omega^2),
\end{aligned}
\tag{22}
$$

or, by eliminating the now useless primes[4], as

$$
\begin{aligned}
&\mathcal{L}_{\mathrm{eff}}(\psi,\partial_{\mu}\psi,\bar{\psi},A^{(r)}_{\mu},\partial_{\nu}A^{(r)}_{\mu},\mathbf{r})\\
&\quad=\mathcal{L}_{0}(\psi,\partial_{\mu}\psi,\bar{\psi},\partial_{\nu}A^{(r)}_{\mu},\mathbf{r})\\
&\qquad+\mathcal{L}_{\mathrm{I}}(\psi,\partial_{i}\psi,\bar{\psi},A^{(r)}_{\mu},\partial_{\nu}A^{(r)}_{\mu},\mathbf{r})
\end{aligned}
\tag{23}
$$

with

$$
\begin{aligned}
&\mathcal{L}_{0}(\psi,\partial_{\mu}\psi,\bar{\psi},\partial_{\nu}A^{(r)}_{\mu},\mathbf{r})\\
&\quad=\bar{\psi}(\mathbf{r},t)\left\{\gamma^{\mu}\left[\mathrm{i}\partial_{\mu}+eA_{\mu}(\mathbf{r})\right]-m\right\}\psi(\mathbf{r},t)\\
&\qquad-\frac{1}{4}F^{(r)}_{\mu\nu}(\mathbf{r},t)F^{(r)\mu\nu}(\mathbf{r},t)
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
&\mathcal{L}_{\mathrm{I}}(\psi,\partial_{i}\psi,\bar{\psi},A^{(r)}_{\mu},\partial_{\nu}A^{(r)}_{\mu},\mathbf{r})\\
&\quad=e\bar{\psi}(\mathbf{r},t)\gamma^{\mu}\psi(\mathbf{r},t)A^{(r)}_{\mu}(\mathbf{r},t)\\
&\qquad+\mathrm{i}\Omega\bar{\psi}(\mathbf{r},t)\gamma^0\mathcal{J}^{(1/2)}_{x}\psi(\mathbf{r},t)\\
&\qquad-\mathrm{i}\Omega\frac{\partial A^{(r)}_{\mu}(\mathbf{r},t)}{\partial t}\mathcal{J}^{(1)}_{x}A^{(r)\mu}(\mathbf{r},t)+O(\Omega^2).
\end{aligned}
\tag{25}
$$

In this way the original time-depending Lagrangian density (1) has been transformed into an effective Lagrangian density that does not depend explicitly on time and that embodies the effects of the rotation of the external magnetic field in the interaction terms proportional to the rotational frequency Ω. We note that the Lagrangian density (23) is just the Lagrangian density of QED in the presence of the external static magnetic field $\mathbf{B}'=(0,0,B)$ plus other extra interaction terms that are proportional to Ω (or to Ω^2).

In order to build the Hamiltonian density we calculate now the momenta conjugate to the Dirac and to the radiation field. From (23)– (25) we obtain

$$
\begin{aligned}
\pi_{\psi}(\mathbf{r},t)&\equiv\frac{\partial\mathcal{L}_{\mathrm{eff}}}{\partial(\partial_0\psi)}\\
&=\mathrm{i}\psi^{\dagger}(\mathbf{r},t),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\boldsymbol{\pi}_{\mathbf{A}^{(r)}}(\mathbf{r},t)&\equiv\frac{\partial\mathcal{L}_{\mathrm{eff}}}{\partial(\partial_0\mathbf{A}^{(r)})}\\
&=\partial_0\mathbf{A}^{(r)}(\mathbf{r},t)+\mathrm{i}\Omega\mathcal{J}^{(1)}_{x}\mathbf{A}^{(r)}(\mathbf{r},t),
\end{aligned}
\tag{27}
$$

and the Hamiltonian density can be written in the form

$$
\begin{aligned}
&\mathcal{H}_{\mathrm{eff}}(\psi,\partial_i\psi,\psi^{\dagger},\mathbf{A}^{(r)},\boldsymbol{\pi}_{\mathbf{A}^{(r)}},\partial_i\mathbf{A}^{(r)},\mathbf{r})\\
&\equiv\pi_{\psi}\partial_0\psi+\boldsymbol{\pi}_{\mathbf{A}^{(r)}}\cdot\partial_0\mathbf{A}^{(r)}\\
&\quad-\mathcal{L}_{\mathrm{eff}}(\psi,\partial_{\mu}\psi,\bar{\psi},A^{(r)}_{\mu},\partial_{\nu}A^{(r)}_{\mu},\mathbf{r})\\
&=\mathcal{H}_0(\psi,\partial_i\psi,\psi^{\dagger},\boldsymbol{\pi}_{\mathbf{A}^{(r)}},\partial_i\mathbf{A}^{(r)},\mathbf{r})\\
&\quad+\mathcal{H}_{\mathrm{I}}(\psi,\partial_i\psi,\psi^{\dagger},\mathbf{A}^{(r)},\boldsymbol{\pi}_{\mathbf{A}^{(r)}},\partial_i\mathbf{A}^{(r)}),
\end{aligned}
\tag{28}
$$

with

$$
\begin{aligned}
&\mathcal{H}_0(\psi,\partial_i\psi,\psi^{\dagger},\boldsymbol{\pi}_{\mathbf{A}^{(r)}},\partial_i\mathbf{A}^{(r)},\mathbf{r})\\
&\quad=\psi^{\dagger}(\mathbf{r},t)\left\{\boldsymbol{\alpha}\cdot\left[-\mathrm{i}\boldsymbol{\nabla}+e\mathbf{A}(\mathbf{r})\right]+\beta m\right\}\psi(\mathbf{r},t)\\
&\qquad+\frac{1}{2}\left\{\left[\boldsymbol{\pi}_{\mathbf{A}^{(r)}}(\mathbf{r},t)\right]^2+\left[\boldsymbol{\nabla}\times\mathbf{A}^{(r)}(\mathbf{r},t)\right]^2\right\}
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
&\mathcal{H}_{\mathrm{I}}(\psi,\partial_i\psi,\psi^{\dagger},\mathbf{A}^{(r)},\boldsymbol{\pi}_{\mathbf{A}^{(r)}},\partial_i\mathbf{A}^{(r)})\\
&\quad=-e\psi^{\dagger}(\mathbf{r},t)\boldsymbol{\alpha}\psi(\mathbf{r},t)\cdot\mathbf{A}^{(r)}(\mathbf{r},t)\\
&\qquad+\mathrm{i}\Omega\psi^{\dagger}(\mathbf{r},t)\mathcal{J}^{(1/2)}_{x}\psi(\mathbf{r},t)\\
&\qquad-\mathrm{i}\Omega\boldsymbol{\pi}_{\mathbf{A}^{(r)}}(\mathbf{r},t)\cdot\mathcal{J}^{(1)}_{x}\mathbf{A}^{(r)}(\mathbf{r},t)+O(\Omega^2).
\end{aligned}
\tag{30}
$$

Now, since the external magnetic field $\mathbf{B}(t)$ has been assumed to be slowly varying in time, we can consider the interaction Hamiltonian density (30) as a small perturbation of the free Hamiltonian density (29) and, at this point, we can use the machinery of the ordinary perturbation theory to calculate the matrix elements corresponding

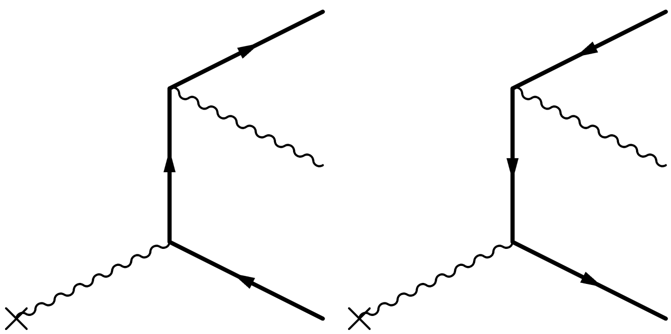
[4] The elimination of the primes on the variables $\mathbf{r}'$ can be safely done only because, the transformation $\mathbf{r}'=\mathbf{r}'(\mathbf{r})$ being a rotation, we have $\mathrm{d}\mathbf{r}'=\mathrm{d}\mathbf{r}$ and $L=\int\mathrm{d}\mathbf{r}\mathcal{L}(\psi,\,\partial_{\mu}\psi,\,\bar{\psi},\,A^{(r)}_{\mu},\,\partial_{\nu}A^{(r)}_{\mu},\,\mathbf{r})=\int\mathrm{d}\mathbf{r}'\mathcal{L}'(\psi',\,\partial'_{\mu}\psi',\bar{\psi}',\,A^{(r)\prime}_{\mu},\,\partial'_{\nu}A^{(r)\prime}_{\mu},\,\mathbf{r}')$ and then the fact that $\mathbf{r}'$ actually depends on time is irrelevant.

to the process under study: the creation of an electron and of a positron together with the emission of a photon by one of the charged particles in the strong slowly rotating magnetic field $\mathbf{B}(t)$. If we neglect all the radiative corrections and take into account only the tree level contributions, the Feynman diagrams accounting for the process are those shown in Fig. 1. The lower interaction vertices represent the creation of the e^-–e^+ pair, while the others represent the electromagnetic emission of a photon by the electron [Fig. 1a] or by the positron [Fig. 1b]. It is clear from the figure that the two processes are not disjoint in time but that the whole process includes both the creation of the pair and the emission of the photon. Now, a pair cannot be created in a constant and uniform magnetic field [10] and, in fact, the interaction vertex with the external field corresponds to the term $\mathrm{i}\Omega\psi^\dagger(\mathbf{r},t)\mathcal{J}_x^{(1/2)}\psi(\mathbf{r},t)$ in $\mathcal{H}_\mathrm{I}(\psi,\partial_i\psi,\psi^\dagger,\mathbf{A}^{(r)},\boldsymbol{\pi}_{\mathbf{A}^{(r)}},\partial_i\mathbf{A}^{(r)})$ proportional to the rotational frequency of the magnetic field. Also, since Ω is assumed to be a small quantity we are allowed to consider only Feynman graphs with one vertex containing the interaction with the external field.

Now, as in ordinary QED, in order to calculate the matrix element of the S-matrix corresponding to the Feynman diagrams in Fig. 1 we quantize the Dirac field and the photon field in the interaction picture. Since, as we have said, the Lagrangian density (24) is the free Lagrangian density of QED in the presence of a uniform and static magnetic field in the z direction with strength B, we already know that the Dirac field can be expanded as [7]

$$\psi(\mathbf{r},t)=\sum_j\left[c_ju_j(\mathbf{r})\exp(-\mathrm{i}w_jt)+d_j^\dagger v_j(\mathbf{r})\exp(\mathrm{i}\tilde{w}_jt)\right], \tag{31}$$

with $j\equiv\{n_d,k,\sigma,n_g\}$ embodying all the quantum numbers and with



(a) (b)

Fig. 1. Tree-level Feynman diagrams of the photon emission by an electron **a** or by a positron **b** created in the presence of the magnetic field $\mathbf{B}(t)$ given in (5). The thick fermion lines indicate that the calculations of the corresponding S-matrix elements have been performed by using the fermion states in the presence of the magnetic field. The vertices representing the interaction with the external magnetic field give a factor $\mathrm{i}\Omega\psi^\dagger(\mathbf{r},t)\mathcal{J}_x^{(1/2)}\psi(\mathbf{r},t)$ (proportional not to the strength of the magnetic field but to its rotational frequency) in the computation of the transition matrix elements

$$w_j=\sqrt{m^2+k^2+eB(2n_d+1+\sigma)}, \tag{32}$$

$$\tilde{w}_j=\sqrt{m^2+k^2+eB(2n_g+1-\sigma)} \tag{33}$$

the Landau energy levels. We are not interested here in the exact form of all electron and positron states $u_j(\mathbf{r})$ and $v_j(\mathbf{r})$ that, anyway, can be found in many textbooks together with the physical meaning of the quantum numbers j and so on [see e.g. [17]]. In fact, from (32) and (33) we see that if the parameter

$$\rho_0=\frac{2eB}{m^2} \tag{34}$$

is much larger than one, there is a class of states, that we called in [7] "transverse ground states" and that are characterized by the quantum numbers $\{n_d=0,k,\sigma=-1,n_g\}$ for the electron and $\{n_d,k,\sigma=+1,n_g=0\}$ for the positron, whose energy, which is independent of B, is much smaller than that of the other states with different quantum numbers. We have seen in [7] that, just for this reason, e^-–e^+ pairs are much more likely created (in the presence of a strong slowly rotating magnetic field) with the electron and the positron in these states. For the same reason, in the present work we assume that all the electrons and the positrons entering the game are in transverse ground states whose explicit expression is [7]:

$$u_{n,k}(\mathbf{r}) \tag{35}$$
$$=\sqrt{\frac{\varepsilon_k+m}{2\varepsilon_k}}\begin{pmatrix}0\\1\\0\\-\frac{k}{\varepsilon_k+m}\end{pmatrix}\varphi_n(x,y)\frac{\exp(\mathrm{i}kz)}{\sqrt{Z}},$$

$$v_{n,k}(\mathbf{r}) \tag{36}$$
$$=\sqrt{\frac{\varepsilon_k+m}{2\varepsilon_k}}\begin{pmatrix}0\\-\frac{k}{\varepsilon_k+m}\\0\\1\end{pmatrix}\varphi_n(x,y)\frac{\exp(-\mathrm{i}kz)}{\sqrt{Z}},$$

with Z the length of the quantization volume in the z direction and

$$\varphi_n(x,y) \tag{37}$$
$$=\sqrt{\frac{1}{\pi n!}\left(\frac{eB}{2}\right)^{n+1}}(x-\mathrm{i}y)^n\exp\left[-\frac{eB}{4}\left(x^2+y^2\right)\right].$$

In order to simplify the notation, the transverse ground states have been labeled only by two indices k and n (since there is no possibility of confusion we omitted the indices "d" and "g" on n_d and n_g) and their energies, that depend only on the longitudinal momentum k, have been indicated as $\varepsilon_k=\sqrt{m^2+k^2}$.

Finally, in the strong magnetic field regime $\rho_0\gg 1$ one is allowed with a good approximation to sum only on the transverse ground states to build the electron propagator $G(\mathbf{r},t,\mathbf{r}',t')$, that is

$$\mathrm{i}G(\mathbf{r},t,\mathbf{r}',t') \tag{38}$$

$$= \sum_{n,k} \{\vartheta(t-t')u_{n,k}(\mathbf{r})\bar{u}_{n,k}(\mathbf{r}')\exp\left[-\mathrm{i}\varepsilon_k(t-t')\right]$$
$$-\vartheta(t'-t)v_{n,k}(\mathbf{r})\bar{v}_{n,k}(\mathbf{r}')\exp\left[\mathrm{i}\varepsilon_k(t-t')\right]\},$$

with $\vartheta(\tau)$ the step function.

We pass now to the second quantization of the radiation field. The presence in the interaction Lagrangian density (25) of terms containing the time derivative of the radiation field would make the quantization procedure very complicated. Nevertheless, we observe that
(1) these additional terms are proportional to the rotational frequency Ω;
(2) the matrix elements that we will calculate are already proportional to Ω through the factor corresponding to the interaction vertex with the external field in Fig. 1.

For these reasons, since we are not interested in higher order corrections in Ω, all the other factors in the matrix elements can be evaluated neglecting the interaction with the external field. In this way, we can quantize the radiation field as it was free and then we have only to expand the vector potential $\mathbf{A}^{(r)}(\mathbf{r},t)$ into the usual plane-wave basis as follows:

$$\mathbf{A}^{(r)}(\mathbf{r},t) = \sum_{\mathbf{q},r} \frac{\mathbf{e}_{\mathbf{q},r}}{\sqrt{2V\omega_q}} \{a_{\mathbf{q},r}\exp[-\mathrm{i}(\omega t-\mathbf{q}\cdot\mathbf{r})] + a^{\dagger}_{\mathbf{q},r}\exp[\mathrm{i}(\omega t-\mathbf{q}\cdot\mathbf{r})]\}, \quad (39)$$

where V is the quantization volume, $\omega = |\mathbf{q}|$ is the photon energy and $\mathbf{e}_{\mathbf{q},r}$ with $r = 1, 2$ are the polarization vectors [18].

At this point we have all the quantities we need to calculate the matrix elements corresponding to the Feynman diagrams in Fig. 1 and this is the first subject of the next section.

3 Calculation of the photon spectrum

By looking at the interaction Hamiltonian density $\mathcal{H}_{\mathrm{I}}(\psi, \partial_i\psi, \psi^{\dagger}, \mathbf{A}^{(r)}, \boldsymbol{\pi}_{\mathbf{A}^{(r)}}, \partial_i\mathbf{A}^{(r)})$ it is clear that the if the final state is the state $|e^-e^+\gamma\rangle \equiv |k_0, n_0; k'_0, n'_0; \mathbf{q}, r\rangle$ (the initial state is obviously the vacuum $|0\rangle$) then the matrix element at time $t > 0$ corresponding to the Feynman diagram in Fig. 1a can be written as[5]

$$S_{k_0,n_0,k'_0,n'_0,\mathbf{q},r}(t) = \int \mathrm{d}\mathbf{r}' \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}\mathbf{r}'' \int_0^t \mathrm{d}t'' u^{\dagger}_{n_0,k_0}(\mathbf{r}')\exp(\mathrm{i}\varepsilon_{k_0}t') \times \frac{e\boldsymbol{\alpha}\cdot\mathbf{e}_{\mathbf{q},r}}{\sqrt{2V\omega}}\exp[\mathrm{i}(\omega t'-\mathbf{q}\cdot\mathbf{r}')]\mathrm{i}G(\mathbf{r}',t',\mathbf{r}'',t'') \times \mathrm{i}\Omega\gamma^0\mathcal{J}_x^{(1/2)\prime\prime}v_{n'_0,k'_0}(\mathbf{r}'')\exp(\mathrm{i}\varepsilon_{k'_0}t''), \quad (40)$$

where we pointed out that, while the electromagnetic interaction between the Dirac field and the radiation field is always present, the external field starts rotating at $t'' = 0$. We also want to stress that the low vertices in Fig. 1 correspond to the term $\mathrm{i}\Omega\gamma^0\mathcal{J}_x^{(1/2)}$ in the expression of the matrix elements. From this point of view, as we have said, the magnetic field being a slowly varying quantity we are allowed to stop the calculations up to first order in Ω and then to consider only Feynman graphs with only one vertex involving the external magnetic field. Instead, since the magnetic field is strong it is fully taken into account in the expression of the propagator $G(\mathbf{r}', t', \mathbf{r}'', t'')$ which is built up by electron and positron states in the presence of the magnetic field [see (38)].

Now, the term in (40) corresponding to the lower vertex in Fig. 1a will be calculated by means of the first order adiabatic perturbation theory [19]. In order to do this, we use the expression (38) of the electron propagator to write the previous matrix element in the more useful form

$$\begin{aligned}
&S_{k_0,n_0,k'_0,n'_0,\mathbf{q},r}(t)\\
&= -\frac{e(\mathbf{e}_{\mathbf{q},r})_z}{\sqrt{2V\omega}}\sum_{n,k}\int_{-\infty}^{t}\mathrm{d}t'\exp[\mathrm{i}(\varepsilon_{k_0}+\omega-\varepsilon_k)t']\\
&\times\int\mathrm{d}\mathbf{r}' u^{\dagger}_{n_0,k_0}(\mathbf{r}')\alpha_z u_{n,k}(\mathbf{r}')\exp(-\mathrm{i}\mathbf{q}\cdot\mathbf{r}')\\
&\times\int_0^{t'}\mathrm{d}t''\exp[\mathrm{i}(\varepsilon_{k'_0}+\varepsilon_k)t'']\\
&\times\int\mathrm{d}\mathbf{r}'' u^{(\mathrm{rot})\dagger}_{n,k}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\
&+\frac{e(\mathbf{e}_{\mathbf{q},r})_z}{\sqrt{2V\omega}}\sum_{n,k}\int_{-\infty}^{t}\mathrm{d}t'\exp[\mathrm{i}(\varepsilon_{k_0}+\omega+\varepsilon_k)t']\\
&\times\int\mathrm{d}\mathbf{r}' u^{\dagger}_{n_0,k_0}(\mathbf{r}')\alpha_z v_{n,k}(\mathbf{r}')\exp(-\mathrm{i}\mathbf{q}\cdot\mathbf{r}')\\
&\times\int_{t'}^{t}\mathrm{d}t''\exp[\mathrm{i}(\varepsilon_{k'_0}-\varepsilon_k)t'']\\
&\times\int\mathrm{d}\mathbf{r}'' v^{(\mathrm{rot})\dagger}_{n,k}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t''),
\end{aligned} \quad (41)$$

where we used the fact that as the transverse ground states (35) and (36) are eigenstates of σ_z, α_x and α_y cannot couple two of them, and where we introduced the "rotated" states

$$u^{(\mathrm{rot})}_{n,k}(\mathbf{r},t) \equiv \exp\left(-\mathrm{i}\mathcal{J}_x^{(1/2)}\Omega t\right)u_{n,k}(\mathbf{r}), \quad (42)$$

$$v^{(\mathrm{rot})}_{n,k}(\mathbf{r},t) \equiv \exp\left(-\mathrm{i}\mathcal{J}_x^{(1/2)}\Omega t\right)v_{n,k}(\mathbf{r}). \quad (43)$$

These states are the instantaneous eigenstates at time t of the one-particle Hamiltonian

$$\mathcal{H}^{(\mathrm{rot})}(\mathbf{r},-\mathrm{i}\boldsymbol{\nabla},t) = \boldsymbol{\alpha}\cdot[-\mathrm{i}\boldsymbol{\nabla}+e\mathbf{A}(\mathbf{r},t)]+\beta m, \quad (44)$$

[5] The Feynman diagram in Fig. 1b represents the emission of the photon by a positron but we can take into account this process by simply multiplying by two the final spectrum of the photons emitted only by an electron.

with $\mathbf{A}(\mathbf{r},t)$ given in (6). In this way, by applying the first order adiabatic perturbation theory [19] we have

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' u^{(\mathrm{rot})\dagger}_{n,k}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=-\frac{1}{\varepsilon_k+\varepsilon_{k'_0}}\int \mathrm{d}\mathbf{r}'' u^{(\mathrm{rot})\dagger}_{n,k}(\mathbf{r}'',t'')\\&\qquad\times\frac{\partial}{\partial t''}\left[\mathcal{H}^{(\mathrm{rot})}(\mathbf{r}'',-\mathrm{i}\boldsymbol{\nabla}'',t'')\right]v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t''),\end{aligned}\tag{45}$$

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' v^{(\mathrm{rot})\dagger}_{n,k}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=-\frac{1}{\varepsilon_{k'_0}-\varepsilon_k}\int \mathrm{d}\mathbf{r}'' v^{(\mathrm{rot})\dagger}_{n,k}(\mathbf{r}'',t'')\\&\qquad\times\frac{\partial}{\partial t''}\left[\mathcal{H}^{(\mathrm{rot})}(\mathbf{r}'',-\mathrm{i}\boldsymbol{\nabla}'',t'')\right]v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'').\end{aligned}\tag{46}$$

These kinds of matrix elements have been calculated in [7]. In particular, by using (47), (48) and (56) of that paper we obtain

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' u^{(\mathrm{rot})\dagger}_{n,k}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=-\frac{1}{\varepsilon_k+\varepsilon_{k'_0}}\int \mathrm{d}\mathbf{r}'' u^{\dagger}_{n,k}(\mathbf{r}'')\frac{e\Omega B}{2}x''\alpha_z v_{n'_0,k'_0}(\mathbf{r}''),\end{aligned}\tag{47}$$

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' v^{(\mathrm{rot})\dagger}_{n,k}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=-\frac{1}{\varepsilon_{k'_0}-\varepsilon_k}\int \mathrm{d}\mathbf{r}'' v^{\dagger}_{n,k}(\mathbf{r}'')\frac{e\Omega B}{2}x''\alpha_z v_{n'_0,k'_0}(\mathbf{r}'').\end{aligned}\tag{48}$$

We observe that in both these matrix elements the integrals on the z variable give a conservation of the longitudinal momentum and then of the energy. This does not cause any problem in the first matrix element, while the second one diverges when $k=k'_0$. For this reason, this particular matrix element will be calculated by writing the left-hand side of (48) as [see (42) and (43)]

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' u^{(\mathrm{rot})\dagger}_{n,k'_0}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=-\mathrm{i}\Omega\int \mathrm{d}\mathbf{r}'' u^{\dagger}_{n,k'_0}(\mathbf{r}'')\mathcal{J}^{(1/2)''}_x v_{n'_0,k'_0}(\mathbf{r}'').\end{aligned}\tag{49}$$

By substituting the explicit expression of the one-particle electron total angular momentum (17), we observe that on the one hand the term $\mathrm{i}z''\partial/\partial y''$ does not contribute because, by performing the integral on z'' from $-Z/2$ to $Z/2$, it vanishes and on the other hand neither the term $\sigma_x/2$ does contribute because the transverse ground states are eigenstates of σ_z. As a result, we have

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' v^{(\mathrm{rot})\dagger}_{n,k'_0}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=-\Omega\int \mathrm{d}\mathbf{r}'' v^{\dagger}_{n,k'_0}(\mathbf{r}'')y''\frac{\partial}{\partial z''}v_{n'_0,k'_0}(\mathbf{r}'').\end{aligned}\tag{50}$$

At this point, if we substitute the expressions (35) and (36) of the transverse ground states we obtain the result that the matrix elements different from zero are

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' u^{(\mathrm{rot})\dagger}_{n'_0-1,-k'_0}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=\frac{m\Omega}{4\varepsilon^2_{k'_0}}\sqrt{\frac{eBn'_0}{2}},\end{aligned}\tag{51}$$

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' u^{(\mathrm{rot})\dagger}_{n'_0+1,-k'_0}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=\frac{m\Omega}{4\varepsilon^2_{k'_0}}\sqrt{\frac{eB(n'_0+1)}{2}},\end{aligned}\tag{52}$$

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' v^{(\mathrm{rot})\dagger}_{n'_0-1,k'_0}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=k'_0\Omega\sqrt{\frac{n'_0}{2eB}},\end{aligned}\tag{53}$$

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}'' v^{(\mathrm{rot})\dagger}_{n'_0+1,k'_0}(\mathbf{r}'',t'')\frac{\partial}{\partial t''}v^{(\mathrm{rot})}_{n'_0,k'_0}(\mathbf{r}'',t'')\\&=-k'_0\Omega\sqrt{\frac{n'_0+1}{2eB}},\end{aligned}\tag{54}$$

where we used the usual expressions of the operators corresponding to the transverse coordinates in terms of the lowering and raising operators related to the quantum numbers n_d and n_g [20]:

$$x''=\frac{1}{2}\sqrt{\frac{2}{eB}}\left(a_g+a^\dagger_g+a_d+a^\dagger_d\right),\tag{55}$$

$$y''=\frac{1}{2i}\sqrt{\frac{2}{eB}}\left(a_g-a^\dagger_g-a_d+a^\dagger_d\right).\tag{56}$$

In this respect, we recall that while the index n stands for n_g in labelling the electron states, it stands for n_d in labelling the positron states.

By inserting the previous matrix elements in (41) and by performing the remaining space-time integrals we obtain the two transition amplitudes[6]

$$\begin{aligned}&S^{(1)}_{k,n,k',n',\mathbf{q},r}(t)\\&=\frac{e\Omega(\mathbf{e}_{\mathbf{q},r})_z}{4}\sqrt{\frac{(\varepsilon_k+m)(\varepsilon_{k'}+m)n'}{\varepsilon_k\varepsilon_{k'}e\omega BV}}I_{n,n'-1,q_x,q_y}\delta_{k+q_z+k',0}\\&\quad\times\left\{\left[\frac{eBm}{8\varepsilon^3_{k'}}\left(\frac{k}{\varepsilon_k+m}-\frac{k'}{\varepsilon_{k'}+m}\right)\frac{1}{\varepsilon_k+\varepsilon_{k'}+\omega}\right.\right.\\&\quad\left.+\frac{k'}{(\varepsilon_k+\varepsilon_{k'}+\omega)^2}\left(1+\frac{k}{\varepsilon_k+m}\frac{k'}{\varepsilon_{k'}+m}\right)\right]\\&\qquad\times\exp[\mathrm{i}(\varepsilon_k+\varepsilon_{k'}+\omega)t]\\&\quad\left.-\frac{eBm}{8\varepsilon^3_{k'}}\left(\frac{k}{\varepsilon_k+m}-\frac{k'}{\varepsilon_{k'}+m}\right)\frac{\exp[\mathrm{i}(\varepsilon_k-\varepsilon_{k'}+\omega)t]}{\varepsilon_k-\varepsilon_{k'}+\omega}\right\}\end{aligned}\tag{57}$$

[6] For notational simplicity, we omitted the now useless index "0" on k_0 and k'_0.

and

$$\begin{aligned}
&S^{(2)}_{k,n,k',n',\mathbf{q},r}(t)\\
&=\frac{e\Omega(\mathbf{e}_{\mathbf{q},r})_z}{4}\sqrt{\frac{(\varepsilon_k+m)(\varepsilon_{k'}+m)(n'+1)}{\varepsilon_k\varepsilon_{k'}e\omega BV}}\\
&\quad\times I_{n,n'+1,q_x,q_y}\delta_{k+q_z+k',0}\\
&\quad\times\left\{\left[\frac{eBm}{8\varepsilon^3_{k'}}\left(\frac{k}{\varepsilon_k+m}-\frac{k'}{\varepsilon_{k'}+m}\right)\frac{1}{\varepsilon_k+\varepsilon_{k'}+\omega}\right.\right.\\
&\quad\left.-\frac{k'}{(\varepsilon_k+\varepsilon_{k'}+\omega)^2}\left(1+\frac{k}{\varepsilon_k+m}\frac{k'}{\varepsilon_{k'}+m}\right)\right]\\
&\quad\times\exp[\mathrm{i}(\varepsilon_k+\varepsilon_{k'}+\omega)t] \qquad (58)\\
&\quad\left.-\frac{eBm}{8\varepsilon^3_{k'}}\left(\frac{k}{\varepsilon_k+m}-\frac{k'}{\varepsilon_{k'}+m}\right)\frac{\exp[\mathrm{i}(\varepsilon_k-\varepsilon_{k'}+\omega)t]}{\varepsilon_k-\varepsilon_{k'}+\omega}\right\}.
\end{aligned}$$

with

$$I_{n,n',q_x,q_y}=\int\mathrm{d}x\mathrm{d}y\varphi^*_n(x,y)\varphi_{n'}(x,y)\exp[-\mathrm{i}(q_xx+q_yy)]. \tag{59}$$

It is worth giving the explicit result of the time integrals in (41):

$$\begin{aligned}
&\int_{-\infty}^{t}\mathrm{d}t'\exp[\mathrm{i}(\varepsilon_k+\omega-\varepsilon_{k'})t']\int_0^{t'}\mathrm{d}t''\exp(2\mathrm{i}\varepsilon_{k'}t'')\\
&=\frac{1}{2\mathrm{i}\varepsilon_{k'}}\left\{\frac{\exp[\mathrm{i}(\varepsilon_k+\varepsilon_{k'}+\omega+\mathrm{i}s)t]}{\mathrm{i}(\varepsilon_k+\varepsilon_{k'}+\omega+\mathrm{i}s)}\right.\\
&\quad\left.-\frac{\exp[\mathrm{i}(\varepsilon_k-\varepsilon_{k'}+\omega+\mathrm{i}s)t]}{\mathrm{i}(\varepsilon_k-\varepsilon_{k'}+\omega+\mathrm{i}s)}\right\} \qquad (60)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{t}\mathrm{d}t'\exp[\mathrm{i}(\varepsilon_k+\omega+\varepsilon_{k'})t']\int_{t'}^{t}\mathrm{d}t''\\
&=\exp[\mathrm{i}(\varepsilon_k+\varepsilon_{k'}+\omega)t]\int_0^{\infty}\mathrm{d}\tau\,\tau\exp[-\mathrm{i}(\varepsilon_k+\varepsilon_{k'}+\omega)\tau]\\
&=-\frac{\exp[\mathrm{i}(\varepsilon_k+\varepsilon_{k'}+\omega-\mathrm{i}s)t]}{(\varepsilon_k+\varepsilon_{k'}+\omega-\mathrm{i}s)^2}. \qquad (61)
\end{aligned}$$

The is terms with $s\to 0^+$ have been added in order to make the integrals convergent. Now, it is obvious that $\varepsilon_k+\varepsilon_{k'}+\omega>0$. Also, because of the overall conservation of the longitudinal momentum $k+q_z+k'=0$, unless we have the trivial case $\mathbf{q}=\mathbf{0}$ it can be shown that $\varepsilon_k-\varepsilon_{k'}+\omega>0$, and all the i$s$ terms can be safely eliminated in the final results in (60) and (61).

The probability that a photon is emitted at time t with momentum between $\mathbf{q}$ and $\mathbf{q}+\mathrm{d}\mathbf{q}$ by an electron or by a positron is obtained by integrating on the quantum numbers of the electron and by multiplying by two:

$$\begin{aligned}
&\mathrm{d}P(\mathbf{q};t)\\
&=2\frac{V\mathrm{d}\mathbf{q}}{(2\pi)^3}\frac{Z}{2\pi}\int\mathrm{d}k\frac{Z}{2\pi}\int\mathrm{d}k' \qquad (62)\\
&\quad\times\sum_{r=1}^{2}\sum_{n,n'=0}^{\infty}\left[\left|S^{(1)}_{n,n',r}(k,k',\mathbf{q};t)\right|^2+\left|S^{(2)}_{n,n',r}(k,k',\mathbf{q};t)\right|^2\right]
\end{aligned}$$

where the limit of large Z and V is understood and all the momenta are intended from now on to be continuous variables. As usual, we are interested in macroscopic times t such that $mt\gg 1$ [7]; then we can neglect the oscillating terms coming from the square modulus of $S^{(1)}_{n,n',r}(k,k',\mathbf{q};t)$ and $S^{(2)}_{n,n',r}(k,k',\mathbf{q};t)$:

$$\begin{aligned}
&\mathrm{d}P(\mathbf{q};t\to\infty)\\
&\sim\frac{Z\mathrm{d}\mathbf{q}}{(2\pi)^4}\frac{e\Omega^2}{8\omega B}\sum_{r=1}^{2}\left|[\mathbf{e}_r(\mathbf{q})]_z\right|^2\\
&\quad\times\sum_{n,n'=0}^{\infty}\left[n'|I_{n,n'-1}(q_x,q_y)|^2+(n'+1)|I_{n,n'+1}(q_x,q_y)|^2\right]\\
&\quad\times\int\mathrm{d}k\frac{[\varepsilon(k)+m][\varepsilon(k')+m]}{\varepsilon(k)\varepsilon(k')}\\
&\quad\times\left\{\left[\frac{eBm}{8\varepsilon^3(k')}\right]^2\left[\frac{k}{\varepsilon(k)+m}-\frac{k'}{\varepsilon(k')+m}\right]^2\right.\\
&\quad\times\left[\frac{1}{[\varepsilon(k)+\varepsilon(k')+\omega]^2}+\frac{1}{[\varepsilon(k)-\varepsilon(k')+\omega]^2}\right]\\
&\quad+\frac{k'^2}{[\varepsilon(k)+\varepsilon(k')+\omega]^4}\\
&\quad\left.\times\left[1+\frac{k}{\varepsilon(k)+m}\frac{k'}{\varepsilon(k')+m}\right]^2\right\}_{k'=-k-q_z}, \qquad (63)
\end{aligned}$$

where we exploited the longitudinal momentum conservation to perform the integration on k'.

We will now calculate separately the sums on the variables r, n and n'. The sum on r is quite trivial and in many textbooks one can find that [18]

$$\sum_{r=1}^{2}\left|[\mathbf{e}_r(\mathbf{q})]_z\right|^2=1-\frac{q_z^2}{\omega^2}=\frac{q_\perp^2}{\omega^2}, \tag{64}$$

where obviously $q_\perp^2=q_x^2+q_y^2$. Concerning the sums on n and n', we will calculate them together with the integrals $I_{n,n'\pm1}(q_x,q_y)$. In fact, we have

$$\begin{aligned}
&\sum_{n,n'=0}^{\infty}\left[n'|I_{n,n'-1}(q_x,q_y)|^2+(n'+1)|I_{n,n'+1}(q_x,q_y)|^2\right]\\
&=\sum_{n,n'=0}^{\infty}(2n'+1)|I_{n,n'}(q_x,q_y)|^2. \qquad (65)
\end{aligned}$$

Now, from (59) and (37) we can write $I_{n,n'}(q_x,q_y)$ as

$$\begin{aligned}
&I_{n,n'}(q_x,q_y)\\
&=\frac{1}{\pi\sqrt{n!n'!}}\int\mathrm{d}\xi\mathrm{d}\eta\,(\xi+\mathrm{i}\eta)^n\,(\xi-\mathrm{i}\eta)^{n'} \qquad (66)
\end{aligned}$$

$$\times \exp\left[-\left(\xi^2+\eta^2\right)\right]\exp\left[-\mathrm{i}\sqrt{\frac{2}{eB}}\left(q_x\xi+q_y\eta\right)\right],$$

where the change of variable

$$\xi=\sqrt{\frac{eB}{2}}x, \tag{67}$$

$$\eta=\sqrt{\frac{eB}{2}}y \tag{68}$$

has been performed. With this expression we can calculate explicitly the sums on n and n'; in fact

$$\begin{aligned}
&\sum_{n,n'=0}^{\infty}(2n'+1)|I_{n,n'}(q_x,q_y)|^2\\
&=\frac{1}{\pi^2}\int \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\xi'\mathrm{d}\eta'\exp\left[-\left(\xi^2+\eta^2+\xi'^2+\eta'^2\right)\right]\\
&\quad\times\exp\left\{\mathrm{i}\sqrt{\frac{2}{eB}}\left[q_x(\xi'-\xi)+q_y(\eta'-\eta)\right]\right\}\\
&\quad\times\sum_{n,n'=0}^{\infty}\frac{2n'+1}{n!n'!}\left(\xi+\mathrm{i}\eta\right)^n\left(\xi-\mathrm{i}\eta\right)^{n'}\\
&\qquad\times\left(\xi'-\mathrm{i}\eta'\right)^n\left(\xi'+\mathrm{i}\eta'\right)^{n'}\\
&=\frac{1}{\pi^2}\int \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\xi'\mathrm{d}\eta'\exp\left[-\left(\xi'-\xi\right)^2-\left(\eta'-\eta\right)^2\right]\\
&\quad\times\exp\left\{\mathrm{i}\sqrt{\frac{2}{eB}}\left[q_x(\xi'-\xi)+q_y\left(\eta'-\eta\right)\right]\right\}\\
&\quad\times\left[2\left(\xi-\mathrm{i}\eta\right)\left(\xi'+\mathrm{i}\eta'\right)+1\right].
\end{aligned}\tag{69}$$

If we now put

$$\xi_\pm=\frac{\xi'\pm\xi}{\sqrt{2}}, \tag{70}$$

$$\eta_\pm=\frac{\eta'\pm\eta}{\sqrt{2}} \tag{71}$$

we have

$$\begin{aligned}
&\sum_{n,n'=0}^{\infty}(2n'+1)|I_{n,n'}(q_x,q_y)|^2\\
&=\frac{1}{\pi^2}\int \mathrm{d}\xi_+\mathrm{d}\xi_-\mathrm{d}\eta_+\mathrm{d}\eta_-\exp\left[-2\left(\xi_-^2+\eta_-^2\right)\right]\\
&\quad\times\exp\left[\mathrm{i}\frac{2}{\sqrt{eB}}\left(q_x\xi_-+q_y\eta_-\right)\right]\\
&\quad\times\{[\xi_+-\xi_--\mathrm{i}(\eta_+-\eta_-)][\xi_++\xi_-+\mathrm{i}(\eta_++\eta_-)]+1\}.
\end{aligned}\tag{72}$$

As we have pointed out in our previous paper [7], the presence of the external non-uniform electric field $\mathbf{E}(\mathbf{r},t)=-\partial\mathbf{A}(\mathbf{r},t)/\partial t$ [see (6)] breaks the translational symmetry in the plane perpendicular to the magnetic field and makes the production probabilities larger and larger in the regions far from the origin. This can also be seen here by observing that the time derivative of the one-particle Hamiltonian (44) in (45) and (46) is proportional to $\mathbf{E}(\mathbf{r},t)$. The consequence is that the integrals on the variables ξ_+ and η_+ in (72) would be diverging. For this reason we will retain in (72) only the dominant terms, that is

$$\begin{aligned}
&\sum_{n,n'=0}^{\infty}(2n'+1)|I_{n,n'}(q_x,q_y)|^2\\
&\sim\frac{1}{\pi^2}\int \mathrm{d}\xi_-\mathrm{d}\eta_-\exp\left[-2\left(\xi_-^2+\eta_-^2\right)\right]\\
&\quad\times\exp\left[\mathrm{i}\frac{2}{\sqrt{eB}}\left(q_x\xi_-+q_y\eta_-\right)\right]\int \mathrm{d}\eta_+\mathrm{d}\xi_+\left(\xi_+^2+\eta_+^2\right).
\end{aligned}\tag{73}$$

Now, by passing to polar coordinates in the ξ_+–η_+ plane we easily obtain [see (67) and (68), and (70) and (71)]

$$\int \mathrm{d}\eta_+\mathrm{d}\xi_+\left(\xi_+^2+\eta_+^2\right)=\frac{\pi}{2}\left(\sqrt{\frac{eB}{2}}R_{\perp M}\right)^4, \tag{74}$$

where $R_{\perp M}$ is the radius of the integration cylinder already introduced in [7] and whose physical meaning will be explained below.[7] Instead, the integrals on the variables ξ_- and η_- are well-known exponential integrals and we only quote the final result:

$$\begin{aligned}
&\sum_{n,n'=0}^{\infty}(2n'+1)|I_{n,n'}(q_x,q_y)|^2\\
&\quad\sim\frac{1}{4}\left(\frac{eB}{2}\right)^2R_{\perp M}^4\exp\left(-\frac{q_\perp^2}{2eB}\right).
\end{aligned}\tag{75}$$

By substituting (64) and (75) in (63) we obtain the following expression of the probability $\mathrm{d}P(\mathbf{q};t\to\infty)$:

$$\begin{aligned}
\mathrm{d}P(\mathbf{q};t\to\infty)&\sim\frac{eB\Omega^2\alpha}{(8\pi)^3}\frac{q_\perp^2\mathrm{d}\mathbf{q}}{\omega^3}ZR_{\perp M}^4\exp\left(-\frac{q_\perp^2}{2eB}\right)\\
&\times\int \mathrm{d}k\frac{[\varepsilon(k)+m][\varepsilon(k')+m]}{\varepsilon(k)\varepsilon(k')}\\
&\times\left\{\left[\frac{eBm}{8\varepsilon^3(k')}\right]^2\left[\frac{k}{\varepsilon(k)+m}-\frac{k'}{\varepsilon(k')+m}\right]^2\right.\\
&\times\left[\frac{1}{[\varepsilon(k)+\varepsilon(k')+\omega]^2}+\frac{1}{[\varepsilon(k)-\varepsilon(k')+\omega]^2}\right]\\
&+\frac{k'^2}{[\varepsilon(k)+\varepsilon(k')+\omega]^4}\\
&\left.\times\left[1+\frac{k}{\varepsilon(k)+m}\frac{k'}{\varepsilon(k')+m}\right]^2\right\}_{k'=-k-q_z},
\end{aligned}\tag{76}$$

where we introduced the fine structure constant $\alpha=e^2/(4\pi)$.

[7] We will see there why performing the limit $R_{\perp M}\to\infty$ would be conceptually wrong.

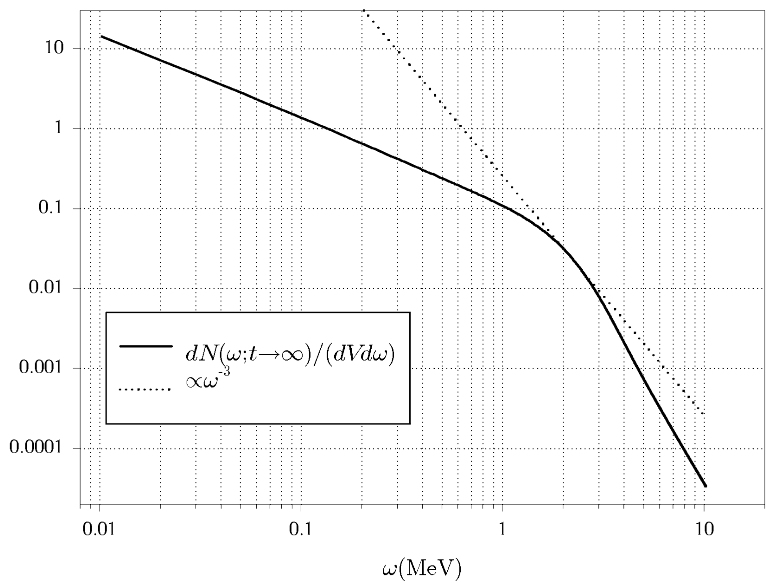


Fig. 2. Photon spectrum $\mathrm{d}N(\omega; t \to \infty)/(\mathrm{d}V\mathrm{d}\omega)$ in arbitrary units. The parameter ρ_0 appearing in (77) has been set equal to 10 corresponding to a magnetic field strength $B = 2.2 \times 10^{14}\,\mathrm{G}$. The dotted curve represents a function proportional to ω^{-3}

Finally, the photon spectrum per unit volume $V = Z\pi R_{\perp M}^2$ is obtained by passing to photon momentum spherical coordinates $\{\omega, \vartheta, \varphi\}$ and integrating on the angular variables. Only the integral on the variable φ is trivial; then by putting $u = \cos\vartheta$ we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}N(\omega; t \to \infty)}{\mathrm{d}V\mathrm{d}\omega} \\
&\sim \frac{\rho_0 m\alpha\omega}{(4\pi)^3}\left(\frac{\Omega R_{\perp M}}{2}\right)^2 \int_{-1}^{1} \mathrm{d}u(1-u^2) \\
&\quad \times \exp\left[-\frac{\omega^2}{m^2\rho_0}(1-u^2)\right] \\
&\quad \times \int_0^\infty \mathrm{d}\lambda \frac{[1+\epsilon(\lambda)][1+\epsilon(\lambda')]}{\epsilon(\lambda)\epsilon(\lambda')} \\
&\quad \times \left\{\left(\frac{\rho_0}{16}\right)^2 \frac{1}{\epsilon^6(\lambda')}\left[\frac{\lambda}{1+\epsilon(\lambda)} - \frac{\lambda'}{1+\epsilon(\lambda')}\right]^2\right. \\
&\quad \times \left[\frac{1}{[\epsilon(\lambda)+\epsilon(\lambda')+\omega/m]^2} + \frac{1}{[\epsilon(\lambda)-\epsilon(\lambda')+\omega/m]^2}\right] \\
&\quad + \frac{\lambda'^2}{[\epsilon(\lambda)+\epsilon(\lambda')+\omega/m]^4} \\
&\quad \left.\times \left[1 + \frac{\lambda}{1+\epsilon(\lambda)}\frac{\lambda'}{1+\epsilon(\lambda')}\right]^2\right\}_{\lambda' = -\lambda - u\omega/m},
\end{aligned} \tag{77}
$$

where we introduced the non-dimensional quantities $\lambda = k/m$ and $\epsilon(\lambda) = \sqrt{1+\lambda^2}$ and where ρ_0 is defined in (34). The presence of the quantity $R_{\perp M}$ in this final result forces us to understand better its physical meaning. Firstly, as we have anticipated below (72), its appearance also in the energy spectrum *per unit volume* (77) is due to the presence of the non-uniform electric field induced by the time variation of $\mathbf{B}(t)$. Now, following the derivation itself of (77) and recalling what we have said in the Introduction, $R_{\perp M}$ can be interpreted as the typical extension of the spatial region within which the magnetic field produced by the astrophysical compact object can be assumed to be uniform. As a consequence the physical assumption about the uniformity of the magnetic field gives an upper limit to the allowed values of $R_{\perp M}$. Also, it can be shown that another limit on $R_{\perp M}$ comes from the fact that we applied the first order adiabatic perturbation theory to calculate the spectrum (77). A too large value of the quantity $\Omega R_{\perp M}$ would make in turn the transition matrix elements (40) too large, and in such a way the first order treatment (in Ω) would be insufficient.

Now, the integrals in (77) cannot be performed analytically; we resort to a numerical integration. Figure 2 shows the photon spectrum (77) in arbitrary units and with $\rho_0 = 10$ corresponding to a magnetic field strength $B = 2.2 \times 10^{14}\,\mathrm{G}$ which is typical in the astrophysical scenario sketched in the Introduction where the fireballs giving rise to gamma-ray bursts are supposed to be produced. Even if, as we have said, the present model is too much simplified to exhaustively describe that physical environment, the spectrum in Fig. 2 has some features qualitatively similar to the corresponding ones of gamma-ray bursts. In fact, the spectrum shows two different behaviors below and above the "break" energy $\omega_\mathrm{b} \sim 1$–$3\,\mathrm{MeV}$[8]. Now, the experimental break energies are typically just below 1 MeV, but there are also cases of gamma-ray bursts with $\omega_\mathrm{b}^{(\mathrm{exp})} > 1\,\mathrm{MeV}$ [21, 22]. On the other hand, the experimental spectra of gamma-ray bursts are very well fitted by a function proportional to ω^{-1} in the low energy region and by a function proportional to $\omega^{-\beta}$ with $\beta \sim 2$–3 in the high energy region [16]. In our case, we see from the figure that the high energy part of the spectrum decreases more rapidly than ω^{-3}. This can be due to the fact that, as we have already pointed out in [8], the production of high energy e^-–e^+ pairs due to a slowly rotating magnetic field is disfavored and this makes the production of high energy

[8] We have checked numerically that the value of ω_b depends on the parameter ρ_0 and also on the strength of the magnetic field (in particular, the lower is ρ_0 the lower is ω_b).

photons less efficient. Instead, concerning the low energy region of the spectrum, we want to conclude by showing analytically that the spectrum (77) goes just as ω^{-1} in the limit $\omega/m \ll 1$. In fact, all the terms in the integrals on u and λ that are finite if calculated at $\omega/m = 0$ give a linear dependence of the spectrum on the photon energy because of the presence of the overall factor proportional to ω in (77). These terms as a result are found to be negligible with respect to the term

$$\frac{\mathrm{d}N(\omega;t\to\infty)}{\mathrm{d}V\mathrm{d}\omega} \overset{\omega/m\ll 1}{\sim} \frac{m\alpha\omega}{4}(\Omega R_{\perp M})^2\left(\frac{\rho_0}{16\pi}\right)^3\int_{-1}^{1}\mathrm{d}u(1-u^2) \times\int_0^\infty \mathrm{d}\lambda\frac{\lambda^2}{(1+\lambda^2)^4}\frac{1}{[\epsilon(\lambda)-\epsilon(\lambda+u\omega/m)+\omega/m]^2}, \tag{78}$$

which is the only one in (77) which diverges in the low energy limit. Now, we will manipulate only the diverging factor $[\epsilon(\lambda)-\epsilon(\lambda+u\omega/m)+\omega/m]^{-2}$ by writing it in the form

$$\frac{1}{[\epsilon(\lambda)-\epsilon(\lambda+u\omega/m)+\omega/m]^2} \simeq\left(\frac{m}{\omega}\right)^2\frac{1+\lambda^2}{\left[\epsilon(\lambda)-\lambda u\right]^2}. \tag{79}$$

By substituting this expression in (78) we finally have

$$\frac{\mathrm{d}N(\omega;t\to\infty)}{\mathrm{d}V\mathrm{d}\omega} \overset{\omega/m\ll 1}{\sim} \frac{\alpha}{4\omega}(\Omega R_{\perp M})^2\left(\frac{\rho_0 m}{16\pi}\right)^3 \times\int_{-1}^{1}\mathrm{d}u(1-u^2)\int_0^\infty \mathrm{d}\lambda\frac{\lambda^2}{(1+\lambda^2)^3}\frac{1}{[\epsilon(\lambda)-\lambda u]^2}, \tag{80}$$

which is the desired result. In fact, since $\epsilon(\lambda)-\lambda u > 0$ in the integration domain, the two integrals are finite and then

$$\frac{\mathrm{d}N(\omega;t\to\infty)}{\mathrm{d}V\mathrm{d}\omega} \overset{\omega/m\ll 1}{\propto} \omega^{-1}. \tag{81}$$

4 Conclusions

In this paper we have studied the electrodynamic process in which a photon is emitted together with an e^-–e^+ pair in the presence of a strong slowly rotating magnetic field. In particular we have calculated the spectrum of the photons emitted by means of this process. We started from the Lagrangian density of QED in the presence of the external rotating magnetic field to build an effective time-independent Lagrangian density that takes into account the rotation of the magnetic field through the presence of interaction terms proportional to the rotational frequency Ω of the field. This form as a result was found to be particularly suitable for our scope, because the astrophysical scenario we imagine to apply to our calculations allowed us to assume the magnetic field to be slowly rotating and then to consider Ω as a small quantity with respect to the microscopic frequency scale m and to the photon frequencies entering the game. In this way, we had the possibility to use the ordinary perturbation theory to calculate the transition matrix elements corresponding to the process under study. As we have seen in Sect. 2, at tree level the process involved second order transition matrix elements in order to account for the creation of the electrons (positrons) and for the photon emission by the electrons (positrons) themselves. As a consequence, the final spectrum (77) was proportional to the fine structure constant α and to the square of the rotational frequency Ω of the external magnetic field.

The strength of the magnetic field has been assumed to be much larger than $B_{\mathrm{cr}} = m^2/e$, and this condition is satisfied only in certain astrophysical systems such as magnetars or massive black holes. From this point of view, the study of the possible electrodynamic processes that can happen in such strong magnetic fields can be submitted to experimental checks. Obviously, these astrophysical environments are actually much more complicated than our theoretical model. Nevertheless, we pointed out that our simple toy model reproduces some qualitative features of the experimental spectra of gamma-ray bursts that are supposed to be originating around massive black holes. Firstly, we have seen a change in the dependence on the photon energy in correspondence with a "break" energy ω_{b}. The value of ω_{b} depends on the magnetic field strength, and it is significant that a realistic value of the magnetic field strength $\sim 10^{14}$ G corresponds to the equally experimental correct values $\omega_{\mathrm{b}} \sim$ 1–3 MeV. Most important, we have also shown analytically that the spectrum we obtained has a linear dependence on the inverse of the photon energy in the low energy region exactly as the spectra of gamma-ray bursts. Instead, we have pointed out that in the region with $\omega > \omega_{\mathrm{b}}$ the theoretical spectrum decreases more rapidly than the experimental spectra and that this can be due to the fact that in our model the production of high energy e^-–e^+ pairs (and then of high energy photons) is not very efficient.

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